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EVOLUTION OF LOCALIZED DISTURBANCES OF A STRATIFIED FLUID WITH VARIABLE BRENT-VAISALA FREQUENCY*

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We construct the approximate expression for Green's function $\Gamma(t, \sqrt{x^2+y^2}, z, z_0)$ of the equation of internal waves in the case of a variable buoyancy frequency-squared

$$N^2(z) = -\frac{9}{\rho_0(t)}\frac{d\rho_0(z)}{dz} = B^2 z,$$

which has, for $z, z_0 > 0$ and $t \to \infty$, the same short-wave asymptotic form as the exact Green's function. This asymptotic form proves to be qualitatively different from the case $N = \text{const}$, considered in detail by many authors, see e.g., /1-10/*. (*Gorodtsov V.A. and Teodorovich E.V., Linear internal waves in an exponentially stratified ideally incompressible fluid, Preprint Inst. Problem Mekhaniki, AN SSSR, Moscow, No.114, 1978.) With $N = \text{const}$ Green's function decreases as $t \to \infty$ as $t^{-1/2}$ and consists of two terms $\Gamma_1 + \Gamma_2$, where Γ_1 oscillates with frequency N, and Γ_2 with frequency $N\cos\theta$ (θ is the angle between the direction from the source to the point of observation and the z axis /4, 5, 8/). For $N = B^2 z$ Green's function oscillates only outside the domain V, given by the equation $r\sqrt{z_-} \leqslant {}^2/_3 \, | z - z_0 |^{3/2}$, where $z_- = \min(z, z_0)$, $r = \sqrt{x^2+y^2}$. Inside V, Green's function does not oscillate but decreases monotonically as t^{-1}. Outside V, the function consists of two terms similar to Γ_1 and Γ_2 for the case $N = \text{const}$.

1. Formulation of the problem. The problem of the propagation and evolution of small disturbances of linear internal waves in an unbounded ideally stratified fluid in the case of localized initial disturbances amounts to constructing Green's function $\Gamma(t, \sqrt{x^2+y^2}, z, z_0)$, i.e., to find, for $t > 0$, the solution of the equation

$$L\Gamma = \frac{\partial^2}{\partial t^2}\Delta\Gamma + N^2(z)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Gamma = 0 \tag{1.1}$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

with the initial conditions at $t = 0$

$$\Delta\Gamma = 0;\ \partial/\partial t\, \Delta\Gamma = \delta(x)\,\delta(y)\,\delta(z - z_0) \tag{1.2}$$

The asymptotic form of Γ as $t \to \infty$ determines the asymptotic form of the wave field in the case of localized initial disturbances.

In the case of constant N [1—10] Green's function Γ is constructed by Fourier's method /4, 5, 8/, and its asymptotic form as $t \to \infty$ in /5/: $\Gamma = \Gamma_-(t) + \Gamma_+(t\,|\cos\theta|)$, where

$$\Gamma_\pm \sim \pm\frac{\sqrt{2}\sin\left(Nt \pm \frac{\pi}{4}\right)}{(N\pi)^{3/2}\rho\sqrt{t}\sin\theta};\quad \rho = \sqrt{x^2+y^2+(z-z_0)^2} \tag{1.3}$$

$$\cos\theta = \frac{z - z_0}{\rho}$$

The first term here is a standing wave, which oscillates with frequency N, while the second has spatial oscillations, whose wavelength tends to zero as $t \to \infty$. The crests of the waves of the second term are cones with vertical z axis, and angle $\alpha = \arccos[\pi(n - {}^1/_4)(Nt)^{-1}]$ which increases with t.

We consider below a medium with the linear dependence

$$N^2(z) = B^2 z \tag{1.4}$$

Since $N^2(z) < 0$ for $z < 0$ and the stratification is unstable, we shall neglect the solutions which increase as $z \to -\infty$. For this medium and a source located at the point $z = z_0 > 0$, $r = \sqrt{x^2+y^2} \neq 0$, we construct the approximate expression u for the Green's function, i.e., the exact solution of (1.1), which satisfies approximately the initial conditions (1.2), up to a function which is analytic for $z_0 > 0$ and any r.

It will be shown that our solution, instead of oscillating everywhere, only oscillates outside a funnel whose equations are

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$$r\sqrt{z_0} < \frac{2}{3}|z - z_0|^{3/2}\ (z < z_0), \quad r\sqrt{z} = \frac{2}{3}|z - z_0|^{3/2}\ (z > z_0) \tag{1.5}$$

Inside the funnel, the field decreases monotonically as $t \to \infty$ as t^{-1}, while outside, it consists of two terms similar to (1.3).

We naturally expect that our assumptions will not affect the qualitative features of the field, i.e., its oscillations, whose wavelength tends to zero as $t \to \infty$.

2. Construction of Green's function. To find Green's function $u(t, r, z, z_0)$, we will construct the approximate expression $\Gamma(t, r, z, z_0)$ for Green's function in the half-space $z < H$ $(H \gg 1)$ with zero boundary condition at $z = H$ and the condition of exponential decrease as $z \to \infty$, after which we find u as the limit of Γ as $H \to \infty$. Since the expression thus obtained for u will nowhere be checked directly, we shall briefly describe the procedure for constructing functions Γ and u, without dwelling on the proof of the approximations employed.

Green's function in the layer $H_0 < z < H_1$ for zero boundary conditions at $z = H_0$, H_1 has the form /11/

$$\Gamma = -\frac{1}{2\pi}\sum_n \int_0^\infty \lambda_n^{-2} J_0(kr)\sin(\omega_n t)\,\varphi_n(k, z)\,\varphi_n(k, z_0)\,\omega_n \frac{dk}{k} \tag{2.1}$$

$$\lambda_n^2 = \int_{H_0}^{H_1} \varphi_n^2(k, z)\,N^2(z)\,dz \tag{2.2}$$

where λ_n is a normalizing factor, $J_0(kr)$ is the Bessel function, and $\omega_n = \omega_n(k)$ and $\varphi_n(k, z)$ are the eigenvalues and eigenfunctions of the boundary value problem

$$\varphi_n'' + k^2/\omega_n^{-2}(N^2(z) - \omega_n^2)\,\varphi_n = 0, \quad \varphi_n(k, H_0) = \varphi_n(kH_1) = 0 \tag{2.3}$$

In our case the role of the lower bound $z = H_0$ is played by the condition for an exponential decrease in φ_n as $z \to -\infty$. Recalling (1.4), we obtain from Eq.(2.3) ($v(\xi)$ is the Airy function)

$$\varphi_n = v(\xi), \quad \xi = \left(\frac{kB}{\omega_n}\right)^{2/3}\left(\frac{\omega_n^2}{B^2} - z\right) \tag{2.4}$$

$$v(\xi) = \frac{1}{2\sqrt{\pi}}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \exp\left[\left(-i\frac{m^3}{3} + m\xi\right)\right] dm \tag{2.5}$$

ε is any positive constant.

To find $\omega_n(k)$, we use the zero boundary condition at $z = H_1$. We then obtain (c_n is the n-th root of the Airy function)

$$\left(\frac{kB}{\omega_n}\right)^{2/3}\left(\frac{\omega_n^2}{B^2} - H_1\right) = c_n \tag{2.6}$$

With $H_1 \gg 1$, only terms with sufficiently large n exist in the sum (2.1). Using the well-known asymptotic form of the root c_n of the Airy function /12/, we obtain from (2.6)

$$\omega_n \simeq \frac{2}{3}\frac{kB}{\pi n} H_1^{3/2} \tag{2.7}$$

We will now find the approximate expression for λ_n when $n \gg 1$. For this, we substitute (2.4) into (2.2) and replace $v(\xi)$ by zero when $\xi > 0$ and by the appropriate asymptotic form when $\xi < 0$. Then,

$$\lambda_n^2 \simeq \frac{B^2}{3} H_1^{3/2}\left(\frac{\omega_n}{kB}\right)^{2/3} \tag{2.8}$$

Substituting (2.8), (2.4) and (2.7) into (2.1) and putting

$$\xi_n = \frac{1}{\omega_n} = \frac{3n\pi}{2kBH_1^{3/2}}, \quad \Delta\xi = \frac{3\pi}{2kBH_1^{3/2}}$$

we obtain for $H_1 \gg 1$ an integral sum, which transforms into an integral as $H_1 \to \infty$. The change of variables $g = kB\xi$, $\omega = (B\xi)^{-1}$ transforms this integral to

$$u = -\frac{1}{\pi^2 B}\int_0^\infty\int_0^\infty g^{1/3}U(g, \omega)\,dg\,d\omega \tag{2.9}$$

$$U(g, \omega) = \sin B\omega t\,U_1(g, \omega), \quad U_1(g, \omega) = J_0(g\omega r)\,v[g^{2/3}(\omega^2 - z)]\,v[g^{2/3}(\omega^2 - z_0)] \tag{2.10}$$

Let us show that u satisfies Eq.(1.1) under condition (1.4). It suffices to show that, given any ω and g, $U(g, \omega)$ satisfies the equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}\right)U+B^2z\left[\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}\right]U=0 \tag{2.11}$$

Hence u, being the superposition (integral with respect to the parameters g and ω with weight $g^{1/3}$) of exact solutions of Eq.(2.11), will likewise be a solution of the equation. Obviously, $u\equiv 0$ for $t=0$. In Sect.3 we find $\partial u/\partial t$ for $t=0$ and show that

$$\partial\Delta u/\partial t\,|_{t=0}=\delta(x)\,\delta(y)\,\delta(z-z_0)+\Psi(r, z) \tag{2.12}$$

where Ψ is a regular function for all r, z.

To facilitate further working, we transform (2.9) for u by writing the product to Airy functions in (2.10) as a product of integrals (2.5) respectively with respect to the variables m and n.

We then make the change of variables $\alpha=m+n, \beta=m-n$, and perform the integration with respect to β. As a result we obtain

$$u=\frac{-e^{i\pi/4}}{4\pi^2\sqrt{\pi B}}\int_0^\infty\int_0^\infty g^{1/3}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\sin\omega Bt J_0(g\omega r)\exp\left[i\left\{\frac{\alpha^3}{12}+g^{2/3}\alpha\left(\omega^2-\frac{z+z_0}{2}\right)-\right.\right.$$
$$\left.\left.g^{4/3}\ \frac{(z-z_0)^2}{4\alpha}\right\}\right]\frac{d\alpha}{\sqrt{\alpha}}\,dg\,d\omega \tag{2.13}$$

3. Check of the initial condition.

We find $\partial u/\partial t\,|_{t=0}$. We differentiate (2.13) with respect to t and put $t=0$. In the result, inasmuch as, with $\operatorname{Im} q>0$ we have /12/

$$\int_0^\infty \xi J_0(p\xi)\exp(iq\xi^2)\,d\xi=\frac{i}{2q}\exp\left(-\frac{ip^2}{4q}\right),$$

we can perform the integration with respect to ω. We then make the change of variables $\tau=\sqrt{g}$, $\alpha=\beta\tau^{2/3}$, and also change the contour of integration with respect to β into a sum of segments of the real axis $|\beta|>\varepsilon$ and a semicircle of radius ε with $\varepsilon\to 0$. We obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=-\frac{e^{3i\pi/4}}{4\pi^{5/2}}\left[\lim_{\varepsilon\to 0} I_1+\lim_{\varepsilon\to 0} I_2\right] \tag{3.1}$$

$$I_1=\int_0^\infty d\tau\left[\int_{-\infty}^{-\varepsilon}+\int_\varepsilon^\infty\right]\exp\{i\tau^2 f(\beta)\}\frac{d\beta}{\beta^{3/2}}$$

$$I_2=-\frac{i}{\sqrt{\varepsilon}}\int_0^\infty d\tau\int_0^\pi d\psi e^{-i\psi/2}\exp\left\{i\tau^2\left[\frac{\varepsilon^3 e^{i\psi^3}}{12}-\frac{\varepsilon\xi e^{i\psi}}{2}-\frac{\rho^2 e^{-i\psi}}{4\varepsilon}\right]\right\}$$

$$f(\beta)=\frac{\beta^3}{12}-\frac{\beta\xi}{2}-\frac{\rho^2}{4\beta},\quad \beta=\varepsilon e^{i\psi},\quad \rho=\sqrt{r^2+(z-z_0)^2},\quad \xi=z+z_0$$

We make the change of variable $\tau=\sqrt{\varepsilon v}$ in the integral I_2 and then obtain

$$\lim_{\varepsilon\to 0} I_2=-i\int_0^\infty dv\int_0^\pi d\psi e^{i\psi/2}\exp\left[-\frac{iv^2\rho^2 e^{i\psi}}{4}\right]=-\frac{\pi^{3/2}\sqrt{i}}{\rho} \tag{3.2}$$

We now evaluate $\lim I_1$ as $\varepsilon\to 0$. Since $\beta^{-3/2}$ is continued on the negative semi-axis through the upper half-plane, we have

$$I_1=\sqrt{2}\,e^{i\pi/4}\int_0^\infty d\tau\int_\varepsilon^\infty\{\cos[\tau^2 f(\beta)]+\sin[\tau^2 f(\beta)]\}\frac{d\beta}{\beta^{3/2}}= \tag{3.3}$$
$$\sqrt{\pi}e^{i\pi/4}\int_{\beta_0}^\infty\frac{d\beta}{\beta^{3/2}\sqrt{f(\beta)}}=\frac{\sqrt{\pi}}{\rho}e^{i\pi/4}\operatorname{arctg}\frac{\rho}{\sqrt{3}\xi},\quad \beta_0=[3\xi+\sqrt{9\xi^2+3\rho^2}]^{1/2}$$

where β_0 is the root of the equation $f(\beta_0)=0$.

As a result, using (3.2), (3.3), we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=+\Phi(r, z),\quad \Phi(r, z)=\frac{1}{4\pi^2\rho}\operatorname{arctg}\frac{\rho}{\sqrt{3}(z+z_0)} \tag{3.4}$$

Hence (2.12) follows, where $\Psi(r, z)=\Delta\Psi(r, z)$ is a regular function for $z_0>0$ and any z, ρ.

4. The asymptotic behaviour of $u(t, r, z, z_0)$ for $t\gg 1$.

By (2.9) and (2.13), we can write u us

$$u(t, r, z, z_0) = -\int_0^\infty \sin B\omega t F(\omega)\, d\omega \tag{4.1}$$

$$F(\omega) = \frac{1}{\pi^2 B}\int_0^\infty g^{1/2} U_1(g, \omega)\, dg = \frac{e^{-i\pi/4}}{4\pi^{5/2} B}\int_0^\infty g^{1/2} I_0(g\omega r)\, dg \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} e^{igf(\beta)} \frac{d\beta}{\sqrt{\beta}} \tag{4.2}$$

It is clear from (4.1) that the asymptotic behaviour of u as $t \to \infty$ for fixed r, z, z_0, is determined, first by the value of $F(\omega)$ at $\omega = 0$, and second, by the singular points of $F(\omega)$, i.e., by the values of ω at which $F(\omega)$ ceases to be analytic.

It can be shown that

$$F(0) = \lim_{\omega \to 0} F(\omega) = \frac{\sqrt{3}}{\pi^2 B (z^2 + zz_0 + z_0^2)}.$$

We now find the singular points of $F(\omega)$ and their contribution to the asymptotic form of u as $t \to \infty$. For this, in the last integral in (4.2) we perform the integration with respect to β on the real axis, then transform from the negative semi-axis to the positive axis by the change $-\beta \to \beta$. The result can be written as

$$F(\omega) = \left.\frac{\partial Q}{\partial \xi}\right|_{\xi=0}, \quad Q = -\frac{1}{2\sqrt{2}\pi^{5/2} B}\int_0^\infty \frac{dg}{\sqrt{g}}\int_0^{2\pi} \cos[g\omega r \sin\theta]\, d\theta\,\cdot$$

$$\int_0^\infty [\sin g(f+\xi) + \cos g(f+\xi)]\frac{d\beta}{\sqrt{\beta}}.$$

Using the relations

$$\int_0^{2\pi} \sin[g\omega r \sin\theta]\, d\theta = 0$$

$$\int_0^{2\pi} \cos(g\omega r \sin\theta)\, d\theta \int_0^\infty \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} g(f+\xi)\ \frac{d\beta}{\sqrt{\beta}} = \int_0^{2\pi} d\theta \int_0^\infty \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} g[f + \xi + \omega r\cos\theta]\frac{d\beta}{\sqrt{\beta}},$$

and performing the integration with respect to g, we can obtain

$$Q = \frac{-1}{2\pi^2 B}\iint_S [\Phi(\beta, \theta, \xi)]^{-1/2}\frac{d\theta\, d\beta}{\sqrt{\beta}}$$

$$\Phi(\beta, \theta, \xi) = \frac{\beta^3}{12} + \beta\left(\omega^2 - \frac{z+z_0}{2}\right) - \frac{(z-z_0)^2}{4\beta} + \omega r\cos\theta + \xi,$$

where the integration is performed only over the part S of the half-strip $0 < \theta < 2\pi, 0 < \beta < \infty$, on which $\Phi(\beta, \theta, \xi) > 0$. The function $Q(0)$, and along with it $\partial Q/\partial\xi|_{\xi=0}$, has singular points at which the analyticity with respect to ω is destroyed only for those ω for which the boundary ∂S of the domain of integration S, given by the equation $\Phi(\beta, \theta, \xi) = 0$, has singular points for $\xi = 0$, i.e., for those ω for which there exists a real solution of the system

$$\Phi(\beta, \theta, 0) = 0, \ \Phi_\beta'(\beta, \theta, 0) = 0, \ \Phi_\theta'(\beta, \theta, 0) = 0 \tag{4.3}$$

From the last of Eqs.(4.3) we have $\cos\theta = \pm 1$. From the second equation we have

$$\beta^2 = m + n \pm 2\sqrt{mn}, \ m = z_0 - \omega^2, \ n = z - \omega^2.$$

In order for β to be real, we must have $\omega^2 < \min(z, z_0)$. We put $z > z_0$. Then the real root β exists only for $\omega^2 < z_0$ and has the form $\beta = \alpha_1\sqrt{m} + \alpha_2\sqrt{n}$, where $\alpha_{1,2} = \pm 1$. Substituting these relations into the first of (4.3) and noting that $\cos\theta = \pm 1$, $\omega > 0$, we arrive at the equation for the singular point ω_1 with $\alpha_1\alpha_2 < 0$ and the singular point ω_2 with $\alpha_1\alpha_2 > 0$:

$$\omega_{1,2} r = \frac{2}{3}[(z - \omega_{1,2}^2)^{3/2} \mp (z_0 - \omega_{1,2}^2)^{3/2}] \tag{4.4}$$

Noting that the left-hand sides of (4.4) are monotonically increasing functions of $\omega_{1,2}$, and that the right-hand sides are decreasing functions, we can show that, when $r\sqrt{z_0} \geqslant {}^2/_3 (z - z_0)^{3/2}$, the singular points ω_1, ω_2 exist and the principal term of the asymptotic behaviour of u is defined by three terms, namely, the contributions of the points $\omega = 0, \omega_1$, and ω_2:

$$u \sim -\frac{\sqrt{3}}{4\pi^2 Bt}\frac{1}{z^2 + zz_0 + z_0} + C_1\frac{\sin(Bt\omega_1 + \pi/4)}{\sqrt{t}} + C_2\frac{\sin(Bt\omega_2 - \pi/4)}{\sqrt{t}} \tag{4.5}$$

where $C_{1,2}$ are functions of the variables r, z, z_0, independent of t.

We can arrive at the result (4.4), (4.5) by a heuristic method, by using in (4.2) the asymptotic form of the Airy functions with $\omega^2 < z, z_0$ and $g \to \infty$, and writing the resulting integrals in the explicit asymptotic form.

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ONE-DIMENSIONAL STABILITY OF DISSIPATIVE COUETTE FLOW*

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Non-stationary one-dimensional flow, and especially the one-dimensional stability of a stationary Couette flow of a viscous incompressible fluid is considered, taking into account dissipative heat, under the assumption that the viscosity decreases fairly rapidly, e.g., exponentially, as the temperature incrases. It is shown that when the fluid is very viscous, the non-stationary plane problem can be reduced to a non-stationary problem of heat transfer in media with heat sources depending non-linearly on temperature. The dependence of the heat sources on temperature in the latter problem differs substantially for different types of boundary conditions in the initial problem. If a tangential stress is specified at the boundary, then the density of the heat sources will depend on temperature locally (such a problem was studied earlier in /1-6/. When the velocities of the boundary planes are given, the density of heat sources will depend on the temperature distribution as a whole, over the volume.

As regards the stationary flow inspected here for stability, it does not always exist, nor is it unique /7-14/. We can utilize the results of /1, 2, 15-19/ by reducing the problem of the existence and uniqueness of such flow to the stationary problem of the temperature distribution in media with heat sources depending non-linearly on temparture.

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